OCR RECOGNISING ACHIEVEMENT

ADVANCED GCE UNIT **4756/01**

MATHEMATICS (MEI)

Further Methods for Advanced Mathematics (FP2)

TUESDAY 16 JANUARY 2007

Morning
Time: 1 hour 30 minutes

Additional materials:
Answer booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions in Section A and **one** question from Section B.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.

ADVICE TO CANDIDATES

- Read each question carefully and make sure you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

Section A (54 marks)

Answer all the questions

1 **(a)** A curve has polar equation $r = ae^{-k\theta}$ for $0 \leqslant \theta \leqslant \pi$, where a and k are positive constants. The points A and B on the curve correspond to $\theta = 0$ and $\theta = \pi$ respectively.

(i) Sketch the curve. [2]

(ii) Find the area of the region enclosed by the curve and the line AB. [4]

(b) Find the exact value of $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{3+4x^2}\,dx$. [5]

(c) **(i)** Find the Maclaurin series for $\tan x$, up to the term in x^3. [4]

(ii) Use this Maclaurin series to show that, when h is small, $\displaystyle\int_h^{4h} \frac{\tan x}{x}\,dx \approx 3h + 7h^3$. [3]

2 **(a)** You are given the complex numbers $w = 3e^{-\frac{1}{12}\pi j}$ and $z = 1 - \sqrt{3}\,j$.

(i) Find the modulus and argument of each of the complex numbers w, z and $\dfrac{w}{z}$. [5]

(ii) Hence write $\dfrac{w}{z}$ in the form $a + bj$, giving the exact values of a and b. [2]

(b) In this part of the question, n is a positive integer and θ is a real number with $0 < \theta < \dfrac{\pi}{n}$.

(i) Express $e^{-\frac{1}{2}j\theta} + e^{\frac{1}{2}j\theta}$ in simplified trigonometric form, and hence, or otherwise, show that

$$1 + e^{j\theta} = 2e^{\frac{1}{2}j\theta}\cos\tfrac{1}{2}\theta.$$

[4]

Series C and S are defined by

$$C = 1 + \binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta + \binom{n}{3}\cos 3\theta + \ldots + \binom{n}{n}\cos n\theta,$$

$$S = \binom{n}{1}\sin\theta + \binom{n}{2}\sin 2\theta + \binom{n}{3}\sin 3\theta + \ldots + \binom{n}{n}\sin n\theta.$$

(ii) Find C and S, and show that $\dfrac{S}{C} = \tan\tfrac{1}{2}n\theta$. [7]

3 Let $\mathbf{P} = \begin{pmatrix} 4 & 2 & k \\ 1 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ (where $k \neq 4$) and $\mathbf{M} = \begin{pmatrix} 2 & -2 & -6 \\ -1 & 3 & 1 \\ 1 & -2 & -2 \end{pmatrix}$.

(i) Find $\mathbf{P}^{-1}$ in terms of k, and show that, when $k = 2$, $\mathbf{P}^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 2 & 4 \\ 4 & -6 & -10 \\ -1 & 2 & 2 \end{pmatrix}$. [6]

(ii) Verify that $\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ are eigenvectors of $\mathbf{M}$, and find the corresponding eigenvalues. [4]

(iii) Show that $\mathbf{M}^n = \begin{pmatrix} 4 & -6 & -10 \\ 2 & -3 & -5 \\ 0 & 0 & 0 \end{pmatrix} + 2^{n-1}\begin{pmatrix} -2 & 4 & 4 \\ -3 & 6 & 6 \\ 1 & -2 & -2 \end{pmatrix}$. [8]

Section B (18 marks)

Answer one question

Option 1: Hyperbolic functions

4 **(i)** Show that $\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right)$. [5]

(ii) Find $\displaystyle\int_{2.5}^{3.9} \frac{1}{\sqrt{4x^2 - 9}}\,dx$, giving your answer in the form $a \ln b$, where a and b are rational numbers. [5]

(iii) There are two points on the curve $y = \dfrac{\cosh x}{2 + \sinh x}$ at which the gradient is $\frac{1}{9}$.

Show that one of these points is $\left(\ln\left(1 + \sqrt{2}\right), \frac{1}{3}\sqrt{2}\right)$, and find the coordinates of the other point, in a similar form. [8]

Option 2: Investigation of curves

This question requires the use of a graphical calculator.

5 Cartesian coordinates (x, y) and polar coordinates (r, θ) are set up in the usual way, with the pole at the origin and the initial line along the positive x-axis, so that $x = r\cos\theta$ and $y = r\sin\theta$.

A curve has polar equation $r = k + \cos\theta$, where k is a constant with $k \geqslant 1$.

(i) Use your graphical calculator to obtain sketches of the curve in the three cases

$$k = 1, k = 1.5 \text{ and } k = 4.$$ [5]

(ii) Name the special feature which the curve has when $k = 1$. [1]

(iii) For each of the three cases, state the number of points on the curve at which the tangent is parallel to the y-axis. [2]

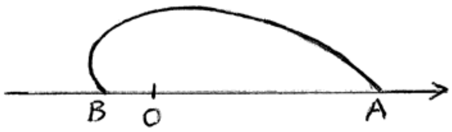
(iv) Express x in terms of k and θ, and find $\frac{\mathrm{d}x}{\mathrm{d}\theta}$. Hence find the range of values of k for which there are just two points on the curve where the tangent is parallel to the y-axis. [4]

The distance between the point (r, θ) on the curve and the point $(1, 0)$ on the x-axis is d.

(v) Use the cosine rule to express d^2 in terms of k and θ, and deduce that $k^2 \leqslant d^2 \leqslant k^2 + 1$. [4]

(vi) Hence show that, when k is large, the shape of the curve is very nearly circular. [2]

Mark Scheme 4756
January 2007

1(a)(i)		B1 B1 **2**	Correct shape for $0 \le \theta \le \frac{1}{2}\pi$ Correct shape for $\frac{1}{2}\pi \le \theta \le \pi$ *Requires decreasing r on at least one axis* *Ignore other values of* θ
(ii)	Area is $\int \frac{1}{2}r^2\,d\theta = \int_0^{\pi} \frac{1}{2}a^2(e^{-k\theta})^2\,d\theta$ $= \left[-\frac{a^2}{4k}e^{-2k\theta} \right]_0^{\pi}$ $= \frac{a^2}{4k}(1-e^{-2k\pi})$	M1 A1 M1 A1 **4**	For $\int (e^{-k\theta})^2\,d\theta$ For a correct integral expression including limits *(may be implied by later work) (Condone reversed limits)* Obtaining a multiple of $e^{-2k\theta}$ as the integral
(b)	$\int_0^{\frac{1}{2}} \frac{1}{3+4x^2}\,dx = \left[\frac{1}{2\sqrt{3}}\arctan\left(\frac{2x}{\sqrt{3}}\right) \right]_0^{\frac{1}{2}}$ $= \frac{1}{2\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right)$ $= \frac{\pi}{12\sqrt{3}}$	M1 A1A1 M1 A1 **5**	For arctan For $\frac{1}{2\sqrt{3}}$ and $\frac{2x}{\sqrt{3}}$ *Dependent on first M1*
	OR Putting $2x = \sqrt{3}\tan\theta$ M1 A1 Integral is $\int_0^{\frac{1}{6}\pi} \frac{1}{2\sqrt{3}}\,d\theta$ A1 M1 $= \frac{\pi}{12\sqrt{3}}$ A1		For any tan substitution For $\int \frac{1}{2\sqrt{3}}\,d\theta$ For changing to limits of θ *Dependent on first M1*
(c)(i)	$f(x) = \tan x, \quad f(0) = 0$ $f'(x) = \sec^2 x, \quad f'(0) = 1$ $f''(x) = 2\sec^2 x\tan x, \quad f''(0) = 0$ $f'''(x) = 2\sec^4 x + 4\sec^2 x\tan^2 x, \quad f'''(0) = 2$ $\tan x = x + \frac{x^3}{3!}(2) + \dots \quad (= x + \frac{1}{3}x^3 + \dots)$	B1 M1 A1 B1 ft **4**	Obtaining $f'''(x)$ For $f''(0)$ and $f'''(0)$ correct ft requires x^3 term and at least one other to be non-zero
(ii)	$\int_h^{4h} \frac{\tan x}{x}\,dx \approx \int_h^{4h} (1+\frac{1}{3}x^2)\,dx$ $= \left[x + \frac{1}{9}x^3 \right]_h^{4h}$ $= (4h + \frac{64}{9}h^3) - (h + \frac{1}{9}h^3)$ $= 3h + 7h^3$	M1 A1 ft A1 ag **3**	Obtaining a polynomial to integrate For $x + \frac{1}{9}x^3$ ft requires at least two non-zero terms

2(a)(i)	$\lvert w \rvert = 3, \quad \arg w = -\frac{1}{12}\pi$ $\lvert z \rvert = 2, \quad \arg z = -\frac{1}{3}\pi$ $\left\lvert \frac{w}{z} \right\rvert = \frac{3}{2}, \quad \arg \frac{w}{z} = (-\frac{1}{12}\pi) - (-\frac{1}{3}\pi) = \frac{1}{4}\pi$	B1 B1B1 B1B1 ft **5**	*Deduct 1 mark if answers given in form $r(\cos\theta + \mathrm{j}\sin\theta)$ but modulus and argument not stated.* Accept degrees and decimal approxs
(ii)	$\frac{w}{z} = \frac{3}{2}(\cos\frac{1}{4}\pi + \mathrm{j}\sin\frac{1}{4}\pi)$ $= \frac{3}{2\sqrt{2}} + \frac{3}{2\sqrt{2}}\mathrm{j}$	M1 A1 **2**	Accept $\sqrt{1.125} + \sqrt{1.125}\,\mathrm{j}$
(b)(i)	$\mathrm{e}^{-\frac{1}{2}\mathrm{j}\theta} + \mathrm{e}^{\frac{1}{2}\mathrm{j}\theta}$ $= (\cos\frac{1}{2}\theta - \mathrm{j}\sin\frac{1}{2}\theta) + (\cos\frac{1}{2}\theta + \mathrm{j}\sin\frac{1}{2}\theta)$ $= 2\cos\frac{1}{2}\theta$	M1 A1	For either bracketed expression
	$1 + \mathrm{e}^{\mathrm{j}\theta} = \mathrm{e}^{\frac{1}{2}\mathrm{j}\theta}(\mathrm{e}^{-\frac{1}{2}\mathrm{j}\theta} + \mathrm{e}^{\frac{1}{2}\mathrm{j}\theta})$ $= \mathrm{e}^{\frac{1}{2}\mathrm{j}\theta}(2\cos\frac{1}{2}\theta)$	M1 A1 ag **4**	
	OR $1 + \mathrm{e}^{\mathrm{j}\theta} = 1 + \cos\theta + \mathrm{j}\sin\theta$ $= 2\cos^2\frac{1}{2}\theta + 2\mathrm{j}\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$ M1 $= 2\cos\frac{1}{2}\theta(\cos\frac{1}{2}\theta + \mathrm{j}\sin\frac{1}{2}\theta)$ $= 2\mathrm{e}^{\frac{1}{2}\mathrm{j}\theta}\cos\frac{1}{2}\theta$ A1		
(ii)	$C + \mathrm{j}S = 1 + \binom{n}{1}\mathrm{e}^{\mathrm{j}\theta} + \binom{n}{2}\mathrm{e}^{2\mathrm{j}\theta} + \dots + \binom{n}{n}\mathrm{e}^{n\mathrm{j}\theta}$ $= (1 + \mathrm{e}^{\mathrm{j}\theta})^n$ $= 2^n \mathrm{e}^{\frac{1}{2}n\theta\mathrm{j}}\cos^n\frac{1}{2}\theta$ $C = 2^n\cos(\frac{1}{2}n\theta)\cos^n\frac{1}{2}\theta$ $S = 2^n\sin(\frac{1}{2}n\theta)\cos^n\frac{1}{2}\theta$ $\frac{S}{C} = \frac{2^n\sin(\frac{1}{2}n\theta)\cos^n\frac{1}{2}\theta}{2^n\cos(\frac{1}{2}n\theta)\cos^n\frac{1}{2}\theta} = \frac{\sin(\frac{1}{2}n\theta)}{\cos(\frac{1}{2}n\theta)} = \tan(\frac{1}{2}n\theta)$	M1 M1A1 M1 A1 A1 B1 ag **7**	Using (i) to obtain a form from which the real and imaginary parts can be written down Allow ft from $C + \mathrm{j}S = \mathrm{e}^{\frac{1}{2}n\theta\mathrm{j}} \times$ any real function of n and θ

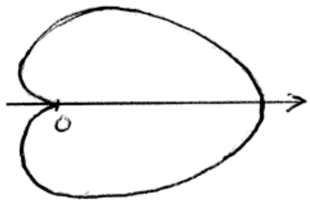
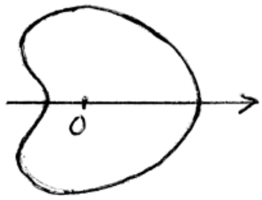
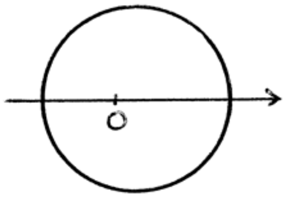
3 (i)	$\det \mathbf{P} = 1(6-k) - 1(4-2)$ $= 4-k$ $\mathbf{P}^{-1} = \frac{1}{4-k}\begin{pmatrix} -1 & 2 & 6-k \\ 4 & -4-k & k-12 \\ -1 & 2 & 2 \end{pmatrix}$ When $k=2$, $\mathbf{P}^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 2 & 4 \\ 4 & -6 & -10 \\ -1 & 2 & 2 \end{pmatrix}$	M1 A1 M1 M1 A1 ft B1 ag **6**	 Evaluating at least three cofactors Fully correct method for inverse Ft from wrong determinant Correctly obtained
(ii)	$\mathbf{M}\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$ $\quad \mathbf{M}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ $\mathbf{M}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ Eigenvalues are 0, 1, 2	M1 A1A1A1 **4**	For one evaluation
	OR M1 Eigenvalues are 0, 1, 2 A2 A1		Obtaining an eigenvalue (e.g. by solving $-\lambda^3 + 3\lambda^2 - 2\lambda = 0$) Give A1 for one correct Verifying given eigenvectors, linking with eigenvalues correctly
(iii)	$\mathbf{M}^n = \begin{pmatrix} 4 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix}\frac{1}{2}\begin{pmatrix} -1 & 2 & 4 \\ 4 & -6 & -10 \\ -1 & 2 & 2 \end{pmatrix}$ $= \frac{1}{2}\begin{pmatrix} 0 & 2 & 2^{n+1} \\ 0 & 1 & 3\times 2^n \\ 0 & 0 & -2^n \end{pmatrix}\begin{pmatrix} -1 & 2 & 4 \\ 4 & -6 & -10 \\ -1 & 2 & 2 \end{pmatrix}$ $= \begin{pmatrix} 4-2^n & -6+2^{n+1} & -10+2^{n+1} \\ 2-3\times 2^{n-1} & -3+3\times 2^n & -5+3\times 2^n \\ 2^{n-1} & -2^n & -2^n \end{pmatrix}$ $= \begin{pmatrix} 4 & -6 & -10 \\ 2 & -3 & -5 \\ 0 & 0 & 0 \end{pmatrix} + 2^{n-1}\begin{pmatrix} -2 & 4 & 4 \\ -3 & 6 & 6 \\ 1 & -2 & -2 \end{pmatrix}$	B1B1 M1A1 B1 ft M1 A1 A1 ag **8**	For $\begin{pmatrix} 4 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix}$ seen (for B2, these must be consistent) For $\mathbf{S}\mathbf{D}^n\mathbf{S}^{-1}$ (M1A0 if order wrong) *or* $\frac{1}{2}\begin{pmatrix} 4 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 4 & -6 & -10 \\ -2^n & 2^{n+1} & 2^{n+1} \end{pmatrix}$ Evaluating product of 3 matrices Any correct form

	OR Prove $\mathbf{M}^n = \mathbf{A} + 2^{n-1}\mathbf{B}$ by induction When $n=1$, $\mathbf{A}+\mathbf{B}=\mathbf{M}$ B1 Assuming $\mathbf{M}^k = \mathbf{A} + 2^{k-1}\mathbf{B}$, $\mathbf{M}^{k+1} = \mathbf{A}\mathbf{M} + 2^{k-1}\mathbf{B}\mathbf{M}$ M1A2 $= \mathbf{A} + 2^{k-1}(2\mathbf{B})$ A1A1 $= \mathbf{A} + 2^k\mathbf{B}$ A1 True for $n=k \Rightarrow$ True for $n=k+1$; hence true for all positive integers *n* A1		or $\mathbf{M}^{k+1} = \mathbf{M}\mathbf{A} + 2^{k-1}\mathbf{M}\mathbf{B}$ *Dependent on previous 7 marks*

4 (i)	If $y = \text{arcosh}\,x$, $x = \cosh y = \frac{1}{2}(e^y + e^{-y})$ $e^{2y} - 2x\,e^y + 1 = 0$ $e^y = \dfrac{2x \pm \sqrt{4x^2 - 4}}{2}$ $= x \pm \sqrt{x^2 - 1}$ Since $y \geq 0$, $e^y \geq 1$, so $e^y = x + \sqrt{x^2 - 1}$ $\text{arcosh}\,x = y = \ln(\,x + \sqrt{x^2 - 1}\,)$	M1 M1 M1 A1 A1 ag **5**	$\frac{1}{2}$ and + must be correct
(ii)	$\displaystyle\int_{2.5}^{3.9} \frac{1}{\sqrt{4x^2 - 9}}\,dx = \left[\,\frac{1}{2}\text{arcosh}\left(\frac{2x}{3}\right)\right]_{2.5}^{3.9}$ $= \frac{1}{2}(\,\text{arcosh}\,2.6 - \text{arcosh}\,\frac{5}{3}\,)$ $= \frac{1}{2}\left(\,\ln(2.6 + \sqrt{2.6^2 - 1}) - \ln(\frac{5}{3} + \sqrt{\frac{25}{9} - 1})\,\right)$ $= \frac{1}{2}(\ln 5 - \ln 3)$ $= \frac{1}{2}\ln\frac{5}{3}$	M1 A1A1 M1 A1 **5**	For arcosh (or any cosh substitution) For $\frac{1}{2}$ and $\frac{2x}{3}$ (or $2x = 3\cosh u$ and $\int \frac{1}{2}\,du$) (or limits of *u* in logarithmic form)
	OR M2 $\left[\,\frac{1}{2}\ln(2x + \sqrt{4x^2 - 9})\,\right]_{2.5}^{3.9}$ A1A1 $= \frac{1}{2}\ln 15 - \frac{1}{2}\ln 9$ $= \frac{1}{2}\ln\frac{5}{3}$ A1		For $\ln(kx + \sqrt{k^2x^2 - ...})$ Give M1 for $\ln(k_1x + \sqrt{k_2{}^2x^2 - ...})$ For $\frac{1}{2}$ and $\ln(2x + \sqrt{4x^2 - 9})$ (or $\ln(x + \sqrt{x^2 - \frac{9}{4}})$)
(iii)	$\dfrac{dy}{dx} = \dfrac{(2 + \sinh x)\sinh x - (\cosh x)(\cosh x)}{(2 + \sinh x)^2}$ $= \dfrac{2\sinh x - 1}{(2 + \sinh x)^2}$ $\dfrac{dy}{dx} = \dfrac{1}{9}$ when $18\sinh x - 9 = (2 + \sinh x)^2$ $\sinh^2 x - 14\sinh x + 13 = 0$ $\sinh x = 1,\ 13$ When $\sinh x = 1$, $\cosh x = \sqrt{2}$, $x = \ln(1 + \sqrt{2})$ Point is $\left(\,\ln(1 + \sqrt{2}),\ \dfrac{\sqrt{2}}{3}\right)$ When $\sinh x = 13$, $\cosh x = \sqrt{170}$, $x = \ln(13 + \sqrt{170})$ Point is $\left(\,\ln(13 + \sqrt{170}),\ \dfrac{\sqrt{170}}{15}\right)$	M1 A1 M1 M1 M1 A1 ag A1A1 **8**	Using quotient rule Any correct form Quadratic in $\sinh x$ (or product of two quadratics in e^x) Solving quadratic to obtain at least one value of $\sinh x$ (or e^x) Obtaining *x* in logarithmic form (must use a correct formula for arsinh) *SR* B1B1 for verifying $y = \frac{1}{3}\sqrt{2}$ and $\dfrac{dy}{dx} = \dfrac{1}{9}$ when $x = \ln(1 + \sqrt{2})$

Alternatives for Q4 (i)

	$\cosh\ln(x+\sqrt{x^2-1}) = \frac{1}{2}(e^{\ln(x+\sqrt{x^2-1})} + e^{-\ln(x+\sqrt{x^2-1})})$ $= \frac{1}{2}(x+\sqrt{x^2-1} + \frac{1}{x+\sqrt{x^2-1}})$ $= \frac{1}{2}(x+\sqrt{x^2-1} + x-\sqrt{x^2-1})$ $= x$ Since $\ln(x+\sqrt{x^2-1}) > 0$, $\operatorname{arcosh} x = \ln(x+\sqrt{x^2-1})$	M1 M1 M1 A1 A1 **5**	
	If $y = \operatorname{arcosh} x$ then $\ln(x+\sqrt{x^2-1}) = \ln(\cosh y + \sqrt{\cosh^2 y - 1})$ $= \ln(\cosh y + \sinh y)$ since $\sinh y > 0$ $= \ln(e^y)$ $= y$	M1 M1 A1 M1 A1 **5**	

5 (i)	$k=1$  $k=1.5$  $k=4$ 	B1 B1 B1 B1 B1 **5**	General shape correct Cusp at O clearly shown General shape correct 'Dimple' correctly shown
(ii)	Cusp	B1 **1**	
(iii)	When $k=1$, there are 3 points When $k=1.5$, there are 4 points When $k=4$, there are 2 points	B2 **2**	Give B1 for two cases correct
(iv)	$x = k\cos\theta + \cos^2\theta$ $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = -k\sin\theta - 2\cos\theta\sin\theta$ $= -\sin\theta(k+2\cos\theta)$ $= 0$ when $\theta = 0,\ \pi,$ or $\cos\theta = -\tfrac{1}{2}k$ For just two points, $k \geq 2$	B1 B1 M1 A1 **4**	Allow $k > 2$
(v)	$d^2 = r^2 + 1^2 - 2r\cos\theta$ $= (k+\cos\theta)^2 + 1 - 2(k+\cos\theta)\cos\theta$ $= k^2 + 1 - \cos^2\theta \quad (= k^2 + \sin^2\theta)$ Since $0 \leq \cos^2\theta \leq 1$, $k^2 \leq d^2 \leq k^2+1$	M1 A1 M1 A1 ag **4**	or $0 \leq \sin^2\theta \leq 1$
(vi)	When *k* is large, $\sqrt{k^2+1} \approx k$, so $d \approx k$ Curve is very nearly a circle, with centre $(1,\ 0)$ and radius *k*	M1 A1 **2**	

4756 - Further Methods for Advanced Mathematics (FP2)

General Comments

There was a wide range of performance on this paper, with about a quarter of the candidates scoring 60 marks or more (out of 72), and about a quarter scoring less than 30 marks. The standard integrals involving inverse trigonometric and hyperbolic functions were handled particularly well, but marks were often lost through carelessness when differentiating trigonometric functions. Many candidates used very long methods to find eigenvalues, and some were unable to complete the paper as a result.
Each of the three questions in Section A had an average mark of about 11 or 12 (out of 18). In Section B, Q.4 (on hyperbolic functions) was chosen by almost all the candidates, and the average mark was about 10.

Comments on Individual Questions

1 **Polar curve and Maclaurin series**

In part (a)(i) the curve was usually sketched correctly, although it sometimes spiralled in the wrong direction and often went through the origin. In part (a)(ii) most candidates used $\int \frac{1}{2} r^2 \, \mathrm{d}\theta$ with the correct limits, although a substantial minority forgot to square *r* even though they had written it down. The work was very often completed correctly, but the integration of $\mathrm{e}^{-2k\theta}$ proved to be surprisingly challenging with factors of *k* going astray and answers such as $-\frac{1}{2k\theta}\mathrm{e}^{-2k\theta}$ or even $\frac{1}{-2k\theta+1}\mathrm{e}^{-2k\theta+1}$ were quite common.
The integral in part (b) was usually found confidently and correctly; the only difficulty was with the factor $\frac{1}{2\sqrt{3}}$ in front of the integral.
In part (c), the method for finding the Maclaurin series was well understood, but the triple differentiation of $\tan x$ very often went wrong. The first derivative was sometimes written as $\frac{1}{\cos^2 x}$ or even $\frac{2}{1+\cos 2x}$ before proceeding, making the work much more difficult than is necessary. Some strong candidates observed that $\mathrm{f}''(x) = 2\mathrm{f}(x)\mathrm{f}'(x)$, and obtained $\mathrm{f}'''(0) = 2$ very efficiently. When the Maclaurin series had been found correctly, the final part (ii) was usually also completed correctly.

2 **Complex numbers**

Part (a) was well answered; most candidates were able to work with modulus and argument correctly, although solutions were quite often spoilt by careless errors (such as an incorrect argument for *z*).
The identity in part (ii) was usually handled successfully.
In part (iii), almost all candidates realised that they should consider $C + \mathrm{j}S$, and there were very many fully correct solutions. However, a fair proportion of candidates failed to recognise the resulting series as binomial, and were determined to use the formulae for a geometric series, thereby losing most of the marks for this part.

3 **Matrices**

In part (i) almost all candidates knew a method for finding the inverse matrix, and the process was very often completed accurately.

In part (ii), it was expected that, for each of the given vectors **e**, the candidates would evaluate **Me** and see that this is a multiple of **e**. Many did this, but a large number of candidates found the characteristic equation, then the eigenvalues, and finally the eigenvectors. This did sometimes yield the correct results, but it must have been very time-consuming. Another common method was to write $(\mathbf{M} - \lambda\mathbf{I})\mathbf{e} = \mathbf{0}$ and use one component to find λ; however, this does not establish that **e** is an eigenvector unless the other two components are checked, and this was rarely done.
Many candidates knew how to answer part (iii) by forming the product $\mathbf{S}\mathbf{D}^n\,\mathbf{S}^{-1}$, although the order of the product was often wrong, and inaccuracies in evaluation frequently prevented the emergence of the given answer.

4 **Hyperbolic functions**

Most candidates knew how to tackle the standard proof in part (i), although few gave the correct reason (arcosh $x \geq 0$) for discarding the other root.

The integral in part (ii) was very often found correctly, although the factor $\frac{1}{2}$ was quite frequently omitted.

In part (iii) the differentiation was usually done correctly. Setting the gradient equal to $\frac{1}{9}$ gives a quadratic in sinh*x* which was often solved correctly; then the logarithmic form of arsinh was usually correctly employed to obtain the values of *x*. Many wrote the gradient in terms of e^x and e^{-x}, and rarely made any further progress; and a common error was to equate the gradient to zero instead of $\frac{1}{9}$. Some made heavy weather of finding the values of *y*; from $\sinh x = 1$ and $x = \ln(1+\sqrt{2})$ they evaluated $\cosh x$ as $\cosh(\ln(1+\sqrt{2}))$ instead of using $\sqrt{1+\sinh^2 x}$.

5 **Investigation of curves**

There were only a few attempts at this question, and all of these scored less than half marks.